International Journal of Solids and Structures 50 (2013) 4161–4172

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Evaluation of the effective elastic moduli of tetragonal fiber-reinforced composites based on Maxwell's concept of equivalent inhomogeneity

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ARTICLE INFO

Article history:
Received 4 March 2013
Received in revised form 30 July 2013
Available online 27 August 2013

Keywords:
Maxwell's methodology
Tetragonal fiber-reinforced composites
Elastic moduli
Circular cylindrical fibers
Fourier series

ABSTRACT

Maxwell's concept of an equivalent inhomogeneity is employed for evaluating the effective elastic properties of tetragonal, fiber-reinforced, unidirectional composites with isotropic phases. The microstructure induced anisotropic effective elastic properties of the material are obtained by comparing the far-field solutions for the problem of a finite cluster of isotropic, circular cylindrical fibers embedded in an infinite isotropic matrix with that for the problem of a single, tetragonal, circular cylindrical equivalent inhomogeneity embedded in the same isotropic matrix. The former solutions precisely account for the interactions between all fibers in the cluster and for their geometrical arrangement. The solutions to several example problems that involve periodic (square arrays) composites demonstrate that the approach adequately captures microstructure induced anisotropy of the materials and provides reasonably accurate estimates of their effective elastic properties.

1. Introduction

The objective of this paper is to generalize the concept of an equivalent inhomogeneity, originally suggested by Maxwell (1873), to allow for accurate evaluation of the overall elastic properties of tetragonal, fiber-reinforced, unidirectional composites with isotropic phases. Maxwell's original concept was suggested to estimate the effective electrical conductivities of a material with spherical particles. The basic idea was to equate the far-field solutions for two problems: one for a finite cluster of spheres embedded in an infinite conducting medium, and the second problem for a single equivalent sphere with unknown electric properties embedded in the same infinite medium. The electrical properties of the equivalent sphere, found from that comparison, were assumed to represent the effective properties of the composite material. The original approach employed a simplifying assumption that at large distances from the cluster all of its particles could be assumed to be located at the same point. Consequently, the problem of the cluster was solved approximately by direct summation of the analytical solutions for a single spherical particle embedded in an infinite matrix. This approach did not account for either the geometrical arrangement of the particles or the interaction between them. The concept has been employed in many publications where it is sometimes labeled as the generalized Maxwell's approach. However, in those publications the word "generalization" meant adaptation of the original Maxwell's approach to a specific physical problem different from that of electric conductivity (e.g. Hasselman and Johnson, 1987; McCartney and Kelly, 2008; McCartney, 2010; Levin et al., 2012); the problem of the cluster was still solved without direct account for the interaction and geometry.

In series of recent papers (Mogilevskaya et al., 2010a,b, 2012a,b; Koroteeva et al., 2010; Mogilevskaya and Crouch, 2013; Pyatigorets and Mogilevskaya, 2011) Maxwell's original approach was modified by solving the cluster problem accounting for the interactions between all of its constituents and for their geometrical arrangement. It was demonstrated that the concept, generalized in that sense, allowed one to accurately evaluate the effective elastic, thermal, and viscoelastic properties of transversely isotropic and particulate composites. In the present paper we go one step further and investigate if the generalized Maxwell approach can capture the microstructure-induced overall anisotropy and can therefore be used for evaluation of the overall behavior of tetragonal, fiber-reinforced, unidirectional composites with isotropic phases.

The material with tetragonal anisotropy was chosen as a case study for two reasons. First of all, materials with such arrangements of fibers are used in various practical applications (e.g. materials with a square arrangement of fibers). Secondly, a number of accurate periodic solutions for such materials have been reported in the literature (e.g. Pobedrya, 1984; Sabina et al., 2002), so the present method could be tested by comparison with those existing solutions.

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http://dx.doi.org/10.1016/j.ijsolstr.2013.08.008

The tetragonal material is characterized by six material constants. To find these constants, several elasticity problems (plane strain, generalized plane strain, and antiplane problems) are formulated and solved both for the cluster and for the equivalent inhomogeneity. The solutions of elasticity problems for clusters have been reported in Mogilevskaya et al. (2008, 2012b) for various cases of loadings at infinity and various conditions on the fiber/matrix interfaces. The general method of solving the plane strain and antiplane problems of a single anisotropic circular inhomogeneity perfectly bonded to an anisotropic matrix is also available, e.g. in Ting (1996), Hwu (2010), and in the references therein. In the present paper we provide closed- form expressions for the specific case of tetragonal symmetry. We believe that the solution of the generalized plane strain problem of a circular anisotropic inhomogeneity embedded in an anisotropic matrix is derived in this paper for the first time.

By comparing the precisely evaluated far-field solutions due to the cluster and those due to the equivalent inhomogeneity with tetragonal symmetry, the six effective constants are expressed in closed forms via the dipole coefficients for the boundary displacements for each fiber. The solutions for the effective properties are compared with the solutions for doubly periodic media.

In the following sections we summarize the problem statement, governing equations, and numerical solution; details are provided in Appendices A and B. The method is illustrated by several examples involving periodic composites with square arrangements of fibers. The last section of the paper summarizes the outcome of the present study and discusses its implications.

2. Problem formulation

Consider a unidirectional fiber-reinforced elastic material with isotropic phases. The non-overlapping long fibers of circular cross-section are perfectly bonded to an isotropic matrix with shear modulus μ_0 and Poisson's ratio ν_0. The fibers with elastic properties (μ_1, ν_1) and radii R_1 are aligned in the x_3 direction and distributed in such a way that the overall properties of the material possess tetragonal symmetry (e.g. as in the case of a square arrangement of fibers shown in Fig. 1).

The constitutive equations for the fibers and the matrix are those of linear elasticity:

$$\sigma_{mn} = 2\mu_k \left[\epsilon_{mn} + \frac{\nu_k}{1 - 2\nu_k} \delta_{mn} \epsilon_{\ell\ell} \right] \tag{1}$$

where $\sigma_{mn}(\epsilon_{mn})$ are the components of the stress (strain) tensor, $m, n = 1, \ldots, 3$, δ_{mn} is the Kronecker's delta, the repeated index implies summation, $k = 0, 1$, and no summation over k is assumed.

The overall properties of the composite material are to be defined.

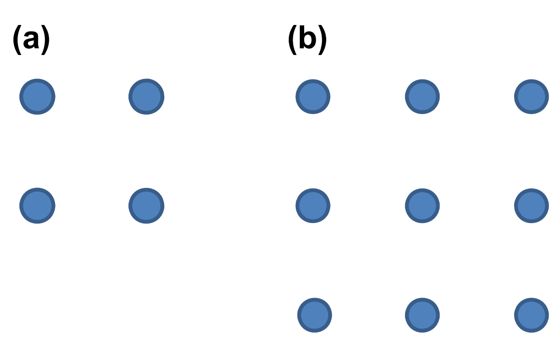

Fig. 1. Square arrays of fibers of equal sizes: (a) four fibers and (b) nine fibers.

2.1. Constitutive equations for the material with tetragonal symmetry

The behavior of the material with tetragonal symmetry is governed by the following constitutive equation

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix} \tag{2}$$

where C_{mn} are elastic constants of the material (contracted notations are used).

We assume that the overall behavior of the composite material under study is governed by Eq. (2) with $C_{mn} = C_{mn}^{ef}$. The constants C_{mn}^{ef} will be found by comparing the far-field solutions for the problem of a finite cluster of N interacting fibers in an infinite matrix (with the properties of the matrix and fibers governed by Eq. (1)) with that of a single equivalent inhomogeneity whose properties are governed by Eq. (2) with $C_{mn} = C_{mn}^{ef}$. The solutions of the three boundary value problems (plane strain, generalized plane strain, and antiplane) are required to find all six unknown constants. The solutions will be given below both for the cluster and the equivalent inhomogeneity.

3. Solutions for the equivalent anisotropic inhomogeneity

Consider an infinite isotropic elastic matrix containing a circular elastic inhomogeneity D_{ef} of radius R_{ef} perfectly bonded to the material matrix (Fig. 2). The boundary of the inhomogeneity is denoted by L_{ef} and its center is assumed to be located at the origin of global Cartesian coordinate system.

3.1. Plane strain problem ($\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$)

The plane strain and antiplane problem of an elliptic anisotropic elastic inhomogeneity perfectly bonded to a dissimilar anisotropic matrix and subjected to a uniform loading at infinity has been solved by several authors (see Section 10.7 in Ting, 1996 for the references and historical background). The problem of a circular inhomogeneity with tetragonal symmetry embedded in an isotropic elastic matrix, needed in this study, is a specific case of that general analysis. The closed-form expressions for the elastic fields related to this case can be obtained using Ting's solution. The

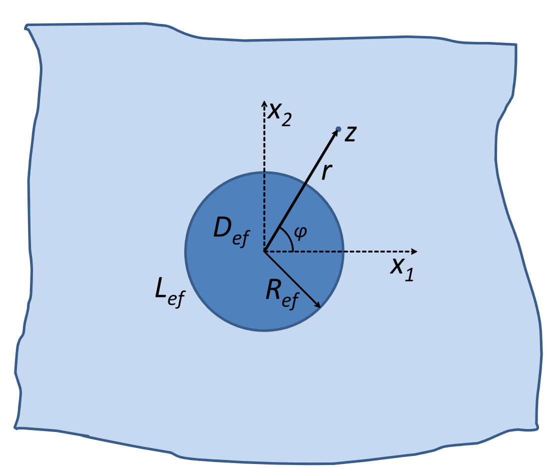

Fig. 2. Equivalent inhomogeneity in an infinite matrix.

details of the derivations are outlined in Appendix A for the case of an inhomogeneity with an arbitrary degree of anisotropy.

The complex displacements $u^m(z)$ at the point $z = r\exp(\imath\varphi)$, $i^2 = -1$ (Fig. 2) of the matrix can be written as the following combination of the complex displacements u^∞ due to the load at infinity and complementary complex displacements u^c induced by the inhomogeneity

$$u^m(z) = u_1^m(z) + iu_2^m(z) = u^\infty + u^c \tag{3}$$

where u_1^m, u_2^m are the components of the displacements in the global Cartesian coordinate system, the superscript "m" ("∞") refers to the matrix material (fields at infinity), and

$$u^\infty = -\frac{\sigma_{22}^\infty - \sigma_{11}^\infty - 2i\sigma_{12}^\infty}{4\mu_0} rg(z) + \frac{\kappa_0 - 1}{2}\frac{\sigma_{22}^\infty + \sigma_{11}^\infty}{4\mu_0} rg^{-1}(z) \tag{4}$$

$$\begin{aligned} u^c = \frac{CR_{ef}^2}{2\mu_0(\kappa_0+1)} \Big\{ & \kappa_0[B_{-2} + \mu_0 A_{-1}]\frac{1}{r}g(z) \\ & + [(\kappa_0 - 1)B_0 + 2\mu_0 A_1]\frac{1}{r}g^{-1}(z) \\ & + [B_{-2} + \mu_0 A_{-1}]\left(1 - \frac{R_{ef}^2}{r^2}\right)\frac{1}{r}g^{-3}(z)\Big\} \end{aligned} \tag{5}$$

where

$$\kappa_0 = 3 - 4\nu_0 \tag{6}$$

$$g(z) = r/z = \exp(-\imath\varphi), \;\; \varphi = \arg z \tag{7}$$

and the combinations of coefficients $C, A_{-1}, A_1, B_0, B_{-2}$ are involved in the representations of complex displacements and tractions at the boundary of the equivalent inhomogeneity (Eqs. (A.11) and (A.12) of Appendix A).

It can be seen from Eq. (5) that the terms in the complementary displacements involving $1/r$ (the leading asymptote, when $r \to \infty$) are represented by three terms of the complex Fourier series at any circle $r = const$.

In the following, the explicit expressions for the coefficients involved in Eq. (5) need to be specified for two types of loading conditions at infinity. These expressions are

(a) $\sigma_{11}^\infty/2\mu_0 = 1$, $\sigma_{22}^\infty = \sigma_{12}^\infty = 0$

$$B_{-2} + \mu_0 A_{-1} = 2\left[\left(C_{12}^{ef}\right)^2 - \left(C_{11}^{ef}\right)^2\right] + 8\mu_0 C_{12}^{ef} + 8\mu_0^2 \tag{8}$$

$$\begin{aligned} (\kappa_0 - 1)B_0 + 2\mu_0 A_1 = \kappa_0(\kappa_0 - 1)&\left[\left(C_{12}^{ef}\right)^2 - \left(C_{11}^{ef}\right)^2\right] \\ & - 2\mu_0(3\kappa_0 - 1)C_{12}^{ef} \\ & + 2\mu_0(\kappa_0+1)C_{11}^{ef} + 8\mu_0^2 \end{aligned} \tag{9}$$

$$\begin{aligned} 1/C = -\frac{2}{\mu_0(\kappa_0+1)}\Big\{ & \kappa_0\left[\left(C_{12}^{ef}\right)^2 - \left(C_{11}^{ef}\right)^2\right] \\ & + 2\mu_0(\kappa_0 - 1)C_{12}^{ef} - 2\mu_0(\kappa_0+1)C_{11}^{ef} - 4\mu_0^2\Big\} \end{aligned} \tag{10}$$

(b) $\sigma_{12}^\infty/2\mu_0 = 1$, $\sigma_{11}^\infty = \sigma_{22}^\infty = 0$

$$B_{-2} + \mu_0 A_{-1} = 2(\mu_0 - C_{44}^{ef}) \tag{11}$$

$$(\kappa_0 - 1)B_0 + 2\mu_0 A_1 = 0 \tag{12}$$

$$1/C = -\imath\frac{\mu_0 + \kappa_0 C_{44}^{ef}}{\mu_0(\kappa_0+1)}. \tag{13}$$

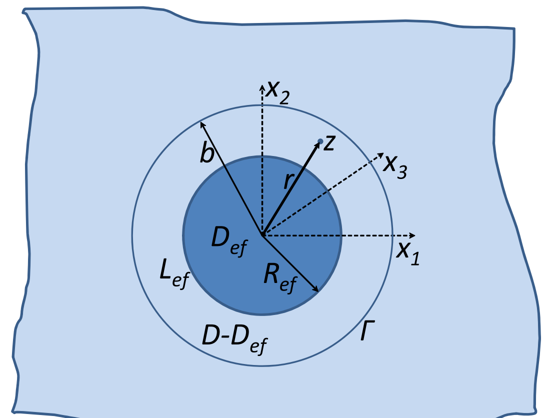

Fig. 3. Circular domain D.

In addition, the following expression for the resultant force F_3 in the Ox_3 direction that acts on a circular domain D ($0 \leqslant r \leqslant b$, $b \geqslant R_{ef}$) with the boundary Γ (Fig. 3), is needed

$$F_3 = \int_D \sigma_{33} dD = \int_{D_{ef}} \sigma_{33}^{in} dD + \int_{D_{ef}-D} \sigma_{33}^m dD \tag{14}$$

where the superscript "in" refers to the inhomogeneity material and

$$\sigma_{33}^m = 2K_0\nu_0(\varepsilon_{11}^m + \varepsilon_{22}^m) \tag{15}$$

$$\sigma_{33}^{in} = C_{13}^{ef}(\varepsilon_{11}^{in} + \varepsilon_{22}^{in}) \tag{16}$$

with $K_0 = \mu_0/(1 - 2\nu_0)$ being two-dimensional plane strain bulk modulus of the matrix.

Substituting Eqs. (15) and (16) into Eq. (14), using the strain–displacement relations, the divergence theorem, and Eqs. (3)–(5), one arrives at the following expression

$$F_3 = 2K_0\nu_0 F_3^\infty + \pi R_{ef}^2 C\left\{(C_{13}^{ef} - C_{13}^m)A_1 + \frac{4\nu_0}{\kappa_0^2 - 1}[(\kappa_0 - 1)B_0 + 2\mu_0 A_1]\right\} \tag{17}$$

with

$$F_3^\infty = \operatorname{Im}\int_\Gamma \bar{u}^\infty d\tau \tag{18}$$

where $\tau = x + \imath y \in \Gamma$, a bar over a symbol denotes complex conjugation, and u^∞ is given by Eq. (4).

Assuming a uniaxial load at infinity ($\sigma_{11}^\infty/2\mu_0 = 1$), the parameters $(\kappa_0 - 1)B_0 + 2\mu_0 A_1$ and C are given by Eqs. (9) and (10), correspondingly, and

$$A_1 = 2\kappa_0(C_{11}^{ef} - C_{12}^{ef}) + 4\mu_0 \tag{19}$$

3.2. Antiplane problem ($\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{12} = 0$)

The procedure outlined in Appendix A also works for the antiplane problem. Following this procedure and assuming that the composite system is subjected to the far-field load $\sigma_{13}^\infty/2\mu_0 = 1$, one obtains the real displacements u_3 of the form

$$u_3 = u_3^\infty + u_3^c = u_3^\infty - 2\frac{C_{55}^{ef} - \mu_0}{C_{55}^{ef} + \mu_0}\frac{R_{ef}^2}{r}\operatorname{Re} g^{-1}(z) \tag{20}$$

where $g^{-1}(z) = 1/g(z)$ and

$$u_3^\infty = 2r\,\mathrm{Re}\,g^{-1}(z) \tag{21}$$

3.3. Generalized plane strain problem ($\varepsilon_{13} = \varepsilon_{23} = 0$, $\varepsilon_{33} = 1$)

In the generalized plane strain problem, the constitutive equations for the inhomogeneity are as follows

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix}C_{11}^{ef} & C_{12}^{ef} & 0\\ C_{12}^{ef} & C_{11}^{ef} & 0\\ 0 & 0 & C_{44}^{ef}\end{bmatrix}\begin{bmatrix}\epsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix} + \begin{bmatrix}C_{13}^{ef}\\ C_{13}^{ef}\\ 0\end{bmatrix} \tag{22}$$

$$\sigma_{33} = C_{13}^{ef}(\varepsilon_{11} + \varepsilon_{22}) + C_{33}^{ef}, \quad \sigma_{13} = \sigma_{23} = 0 \tag{23}$$

and the constitutive equations for the matrix are

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix}K_0 + \mu_0 & K_0 - \mu_0 & 0\\ K_0 - \mu & K_0 + \mu_0 & 0\\ 0 & 0 & \mu_0\end{bmatrix}\begin{bmatrix}\epsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix} + 2K_0\nu_0\begin{bmatrix}1\\ 1\\ 0\end{bmatrix} \tag{24}$$

$$\sigma_{33} = 2K_0\nu_0(\varepsilon_{11} + \varepsilon_{22}) + 2K_0(1 - \nu_0), \quad \sigma_{13} = \sigma_{23} = 0 \tag{25}$$

Assuming zero stresses at infinity, one can re-formulate the generalized plane strain problem of a perfectly bonded inhomogeneity in an infinite matrix as a standard plane strain problem (see Section 4.2 in Mogilevskaya et al., 2012b, for the case of an isotropic inhomogeneity). In the standard plane strain problem the composite system is subjected to the biaxial stresses at infinity

$$\sigma_{11}^\infty = \sigma_{22}^\infty = -2K_0\nu_0 \tag{26}$$

while satisfying the conditions of displacement continuity at the inhomogeneity-matrix interface and the following jump conditions for tractions

$$t^{in} - t^m = -[C_{13}^{ef} - 2K_0\nu_0]g_0^{-1}(z) \tag{27}$$

where $g_0(z) = R_{ef}/z$, $t^{in} = t_1^{in} + it_2^{in}$, t_1^{in}, t_2^{in} are the components of the tractions in the global Cartesian coordinate system on the inhomogeneity side whereas t^m is defined similarly for the matrix side.

The solution of the problem is outlined in Appendix B for the case of an inhomogeneity with arbitrary degree of anisotropy. To the best of our knowledge, this solution is a new one; the previously available solutions for an imperfectly bonded anisotropic inhomogeneity dealt with the interface conditions of traction continuity and a specific form of jump in displacements (e.g. Ting, 2009; Wang, 2009).

Following the procedure outlined in Appendix B, one can again express the complex displacements at the point $z = r\exp(\imath\varphi)$ (Fig. 2) of the matrix by Eqs. (3)–(5) but with the coefficients given by the following expressions:

$$B_{-2} + \mu_0 A_{-1} = 0 \tag{28}$$

$$\begin{aligned}(\kappa_0 - 1)B_0 + 2\mu_0 A_1 = (\kappa_0 - 3)\Big\{\kappa_0\Big[\left(C_{12}^{ef}\right)^2 - \left(C_{11}^{ef}\right)^2\Big]\\ -2\mu_0\left(C_{12}^{ef} + C_{11}^{ef}\right)\Big\} + 4\Big[\kappa_0\left(C_{12}^{ef} - C_{11}^{ef}\right) - 2\mu_0\Big]\end{aligned} \tag{29}$$

and stresses given by Eq. (26).

Knowing the displacements and following the procedure outlined in Section 3.1, one can find the following expression for the resultant force F_3 of Eq. (14):

$$\begin{aligned}F_3 = \pi R_{ef}^2\Big\{(C_{13}^{ef} - 2K_0\nu_0)CA_1 + \frac{4\nu_0}{\kappa_0^2 - 1}C[(\kappa_0 - 1)B_0 + 2\mu_0 A_1]\\ + C_{33}^{ef} + 2\mu_0(1 + \nu_0)\left(\frac{b^2}{R_{ef}^2} - 1\right) - 4K_0\nu_0^2\Big\}\end{aligned} \tag{30}$$

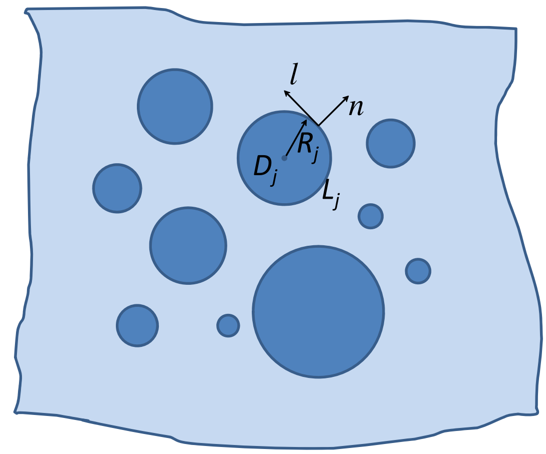

Fig. 4. Cluster of N fibers in an infinite matrix.

4. The solutions for the cluster problem

The solutions of the plane strain, generalized plane strain, and antiplane problems for the cluster of N circular isotropic inhomogeneities (with the properties of those for the fibers) perfectly bonded to an isotropic matrix and subjected to an uniform loading at infinity (Fig. 4) are described in detail in Mogilevskaya et al. (2012b).

The series expansions of complex displacements at the point $z = r\exp\imath\varphi$ of the matrix (for the plane strain and generalized plane strain problems) and the expansions of real displacements at the same point (for the antiplane problem) are given in Appendix B of Mogilevskaya et al. (2012b).

The coefficients in those series are the complex functions of the point z, the elastic properties of the matrix and the fibers, the geometrical parameters of the fibers (radii, locations of the centers), and the complex coefficients in the following Fourier series expansions of the complex displacements at the boundary of the p-th circle:

$$u^p = \sum_{k=1}^{\infty} A_{-kp}g_p^k(z) + \sum_{k=1}^{\infty} A_{kp}g_p^{-k}(z) \tag{31}$$

$$u_3^p + i\omega^p = \sum_{k=1}^{\infty} B_{-kp}g_p^k(z) + \sum_{k=1}^{\infty} B_{kp}g_p^{-k}(z) \tag{32}$$

where u^p are the complex displacements for the plane strain and generalized plane strain problems (different for both problems), $u_3^p + i\omega^p$ is an analytical function whose real part represents the displacement for the antiplane problem, $g_p(z) = R_1/(z - z_p)$, with z_p being the center of the p-th circle.

The unknown coefficients A_{-kp} and A_{kp} (different for the plane strain and generalized plane strain problems) and coefficients B_{-kp}, B_{kp} can be found from the infinite systems of linear equations that are obtained from the interface conditions for each circle and using the orthogonality properties of Fourier series. The procedure is outlined in detail in Appendix A of Mogilevskaya et al. (2012b).

In the following subsections, the expansions for the displacements and resultant forces due to the cluster are presented to compare them with the corresponding expressions of Eqs. (5), (17), (20), and (30) for the equivalent inhomogeneity problems. The expansions due to the cluster are presented in truncated forms in which only the terms involved in leading asymptote $1/r$ are retained. The complete expansions are given in Appendix B of Mogilevskaya et al. (2012b).

4.1. Plane strain problem ($\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$)

The complex series for the complex complementary displacements (induced by the cluster of inhomogeneities) at the point $z = r\exp(\imath\varphi)$ can be written as follows

$$u^c = \frac{1}{\kappa_0+1}\frac{R_1^2}{r}\sum_{p=1}^{N}\left\{\kappa_0(1-\mu_1/\mu_0)\frac{A_{-1p}}{R_1}g(z) + 2(1-K_1/K_0)\mathrm{Re}\frac{A_{1p}}{R_1}g^{-1}(z) + (1-\mu_1/\mu_0)\frac{\overline{A}_{-1p}}{R_1}g^{-3}(z)\right\} + O\left(\frac{1}{r^2}\right) \quad (33)$$

where $K_1 = \mu_1/(1-2\nu_1)$ is two-dimensional plane strain bulk modulus of the fibers.

The counterpart of Eq. (17) for the cluster is

$$F_3 = F_3^\infty + 2\pi R_1^2\sum_{p=1}^{N}\left\{\mathrm{Re}\frac{A_{1p}}{R_1}\left[2(K_1\nu_1 - K_0\nu_0) - \frac{\nu_0}{1-\nu_0}(K_1-K_0)\right]\right\} \quad (34)$$

where F_3^∞ is given by Eq. (18) in which the displacements are obtained from Eq. (4).

4.2. Antiplane problem ($\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{12} = 0$)

The series for the real complementary displacements at the point $z = r\exp(\imath\varphi)$ induced by the cluster of inhomogeneities is

$$u_3^c = \frac{R_1^2}{r}\sum_{p=1}^{N}\mathrm{Re}\left[\frac{B_{-1p}}{R_1}g(z)\right] + O\left(\frac{1}{r^2}\right) \quad (35)$$

4.3. Generalized plane strain problem ($\varepsilon_{13} = \varepsilon_{23} = 0$, $\varepsilon_{33} = 1$)

The counterpart of Eq. (30) for the cluster is

$$F_3 = \pi R_1^2\sum_{p=1}^{N}\left\{2\,\mathrm{Re}\frac{A_{1p}}{R_1}\left[2(K_1\nu_1 - K_0\nu_0) - \frac{\nu_0}{1-\nu_0}(K_1-K_0)\right] + 2[K_1(1-\nu_1) - K_0(1-\nu_0)] - \frac{2\nu_0}{1-\nu_0}(K_1\nu_1 - K_0\nu_0) + 2\frac{b^2}{R_1^2}\mu_0(1+\nu_0)\right\} \quad (36)$$

5. Effective stiffness of the composite using generalized Maxwell's approach

The generalized Maxwell's concept of equivalent inhomogeneity implies that the effective stiffness tensor of the composite can be obtained by comparing the far-field asymptotic behavior of the fields obtained in Sections 3 and 4. Below we follow Maxwell (1873) and define the radius R_{ef} of the equivalent inhomogeneity as

$$R_{ef}^3 = NR_1^2/c \quad (37)$$

This definition preserves the volume fraction c of the inhomogeneities in the cluster.

Taking the load in the following form

$$\sigma_{11}^\infty/2\mu_0 = 1, \quad \sigma_{22}^\infty = \sigma_{12}^\infty = 0 \quad (38)$$

and using the notations

$$\Upsilon_1 = -\frac{4}{\kappa_0+1}(1-\mu_1/\mu_0)c\frac{1}{N}\sum_{p=1}^{N}\mathrm{Re}\frac{A_{-1p}}{R_1} \quad (39)$$

$$\Upsilon_2 = -\frac{8}{\kappa_0+1}(1-K_1/K_0)c\frac{1}{N}\sum_{p=1}^{N}\mathrm{Re}\frac{A_{1p}}{R_1} \quad (40)$$

the following system of equations can be obtained by equating the expressions of Eqs. (5) and (33) that contain $(1/r)g(z)$, $(1/r)g^{-1}(z)$, $(1/r)g^{-3}(z)$:

$$\frac{2\left[\left(C_{12}^{ef}\right)^2 - \left(C_{11}^{ef}\right)^2\right] + 8\mu_0C_{12}^{ef} + 8\mu_0^2}{\kappa_0\left[\left(C_{12}^{ef}\right)^2 - \left(C_{11}^{ef}\right)^2\right] + 2\mu_0(\kappa_0-1)C_{12}^{ef} - 2\mu_0(\kappa_0+1)C_{11}^{ef} - 4\mu_0^2} = \Upsilon_1 \quad (41)$$

$$\frac{\kappa_0(\kappa_0-1)\left[\left(C_{12}^{ef}\right)^2 - \left(C_{11}^{ef}\right)^2\right] + 2\mu_0(1-3\kappa_0)C_{12}^{ef} + 2\mu_0(\kappa_0+1)C_{11}^{ef} + 8\mu_0^2}{\kappa_0\left[\left(C_{12}^{ef}\right)^2 - \left(C_{11}^{ef}\right)^2\right] + 2\mu_0(\kappa_0-1)C_{12}^{ef} - 2\mu_0(\kappa_0+1)C_{11}^{ef} - 4\mu_0^2} = \Upsilon_2 \quad (42)$$

The above procedure implies that the solution for the cluster has a special structure that leads to the vanishing of the imaginary part of the complex sum of the coefficients $\sum_{p=1}^{N}A_{-1p}$. This observation is supported by the numerical experiments of Section 7.

By solving this system, one can find the expressions for the following combinations of effective coefficients C_{11}^{ef}, C_{12}^{ef}:

$$\frac{C_{11}^{ef}+C_{12}^{ef}}{2} = \mu_0\frac{2+\Upsilon_2}{\kappa_0-1-\Upsilon_2} \quad (43)$$

$$\frac{C_{11}^{ef}-C_{12}^{ef}}{2} = \mu_0\frac{2+\Upsilon_1}{2-\Upsilon_1\kappa_0} \quad (44)$$

Comparing Eqs. (17) and (34) for the resultant force, one gets

$$C_{13}^{ef} = \frac{4}{\kappa_0+1}\left(1+\frac{C_{12}^{ef}+C_{11}^{ef}}{2\mu_0}\right)\left\{\mu_0\nu_0 + [2(K_1\nu_1 - K_0\nu_0) - \frac{\nu_0}{1-\nu_0}(K_1-K_0)]c\frac{1}{N}\sum_{p=1}^{N}\mathrm{Re}\frac{A_{1p}}{R_1}\right\} \quad (45)$$

Taking the shear load in the following form

$$\sigma_{12}^\infty/2\mu_0 = 1, \quad \sigma_{22}^\infty = \sigma_{11}^\infty = 0 \quad (46)$$

and comparing Eq. (5) with its counterpart, Eq. (33), for the cluster, one obtains

$$\frac{\mu_0 - C_{44}^{ef}}{\mu_0 + \kappa_0C_{44}^{ef}} = \frac{1}{\kappa_0+1}(1-\mu_1/\mu_0)c\frac{1}{N}\sum_{p=1}^{N}\mathrm{Im}\frac{A_{-1p}}{R_1} \quad (47)$$

where the coefficients A_{-1p} are different from those involved in Eq. (39) as they were obtained under the conditions of a different (shear) load at infinity.

The comparisons of the corresponding terms of Eqs. (20) and (35) for the antiplane problems yields

$$\frac{\mu_0 - C_{55}^{ef}}{\mu_0 + C_{55}^{ef}} = c\frac{1}{N}\sum_{p=1}^{N}\mathrm{Re}\frac{B_{-1p}}{R_1} \quad (48)$$

and, finally, the comparisons of the corresponding terms of Eqs. (30) and (36) gives

$$C_{33}^{ef} = 2\mu_0(1+\nu_0) + 2c\left[K_1(1-\nu_1) - K_0(1-\nu_0) - \frac{\nu_0}{1-\nu_0}(K_1-K_0)\right] + 4c\left[K_1\nu_1 - K_0\nu_0 - \frac{\nu_0}{2(1-\nu_0)}(K_1-K_0)\right]\frac{1}{N}\sum_{p=1}^{N}\mathrm{Re}\frac{A_{1p}}{R_1} + C_{13}^{ef}\frac{C_{13}^{ef}+2\mu_0\nu_0}{0.5(C_{11}^{ef}+C_{12}^{ef})+\mu_0} \quad (49)$$

where the coefficients A_{1p} are obtained from the solution of the generalized plane strain problem, and the effective constants C_{11}^{ef}, C_{12}^{ef}, C_{13}^{ef} are found from Eqs. (43)–(45).

The numerical procedure that utilizes the generalized Maxwell's concept includes the following steps:

- identification of the cluster of N inhomogeneities that adequately represents the composite material in question
- identification of the radius R_{ef} of the equivalent inhomogeneity
- solution of the plane strain, generalized plane strain, and antiplane problems for the cluster to obtain coefficients involved in Eqs. (39), (40), (45) and (47)–(49)
- determination of the effective stiffness coefficients C_{mn}^{ef} from Eqs. (43)–(49).

Due to the fact that the coefficients for the cluster are obtained from the system of equations that has the same structure as the system for the transversely isotropic case (Appendix A in Mogilevskaya et al., 2012b), one can prove (see Section 7 of Mogilevskaya et al., 2012b for the details) that the effective stiffness coefficients of Eqs. (43)–(49) exactly satisfy the universal microstructure-independent relations (e.g., Milton, 2002; Torquato, 2002).

6. Non-interacting estimates

It can be seen from Eqs. (43)–(49) that the overall stiffness coefficients are expressed in terms of the dipole coefficients for the boundary displacements for each fiber. In case when the interactions between the inhomogeneities are not taken into account, the dipole coefficients in each series of Eqs. (31) and (32) are the same for every p, and the expressions given by Eqs. (43)–(49) are the same as for the case when $N = 1$. The dipole coefficients for the latter case are (Appendix A, Mogilevskaya et al., 2012b)

- plane strain (uniaxial load $\sigma_{11}^{\infty}/2\mu = 1$)

$$\frac{1}{N}\sum_{p=1}^{N}\mathrm{Re}\frac{A_{1p}}{R_1} = \frac{(\kappa_0+1)\mu_0}{4(K_1+\mu_0)} \tag{50}$$

$$\frac{1}{N}\sum_{p=1}^{N}\mathrm{Re}\frac{A_{-1p}}{R_1} = \frac{(\kappa_0+1)\mu_0}{2(\kappa_0\mu_1+\mu_0)} \tag{51}$$

- plane strain (shear load $\sigma_{12}^{\infty}/2\mu = 1$)

$$\frac{1}{N}\sum_{p=1}^{N}\mathrm{Im}\frac{A_{-1p}}{R_1} = \frac{(\kappa_0+1)\mu_0}{\kappa_0\mu_1+\mu_0} \tag{52}$$

- antiplane strain

$$\frac{1}{N}\sum_{p=1}^{N}\mathrm{Re}\frac{B_{-1p}}{R_1} = \frac{\mu_0-\mu_1}{\mu_0+\mu_1} \tag{53}$$

generalized plane strain

$$\frac{1}{N}\sum_{p=1}^{N}\mathrm{Re}\frac{A_{1p}}{R_1} = \frac{(\kappa_0-3)\mu_0+(\kappa_1-3)K_1}{4(\mu_0+K_1)} \tag{54}$$

Table 1
Normalized effective moduli $(C_{11}^{eff}+C_{12}^{eff})/2K_0$ and $(C_{11}^{eff}-C_{12}^{eff})/2\mu_0$.

c	$(C_{11}^{eff}+C_{12}^{eff})/2K_0$				$(C_{11}^{eff}-C_{12}^{eff})/2\mu_0$			
	Eq. (55), non-int.	$N = 4$	$N = 9$	RUC	Eq. (56), non-int.	$N = 4$	$N = 9$	RUC
0.05	1.064	1.064	1.064	1.064	1.089	1.092	1.093	1.094
0.10	1.134	1.134	1.134	1.134	1.187	1.201	1.205	1.210
0.15	1.213	1.213	1.213	1.213	1.298	1.330	1.341	1.354
0.20	1.302	1.302	1.302	1.302	1.421	1.485	1.508	1.534
0.25	1.402	1.403	1.403	1.402	1.562	1.672	1.713	1.758
0.30	1.517	1.518	1.518	1.517	1.722	1.900	1.969	2.041
0.35	1.649	1.651	1.652	1.651	1.906	2.182	2.292	2.400
0.40	1.802	1.808	1.809	1.806	2.122	2.534	2.709	2.861
0.45	1.983	1.994	1.998	1.993	2.376	2.984	3.260	3.464
0.50	2.200	2.221	2.229	2.222	2.680	3.574	4.015	4.269
0.55	2.464	2.506	2.522	2.515	3.052	4.376	5.108	5.378
0.60	2.792	2.879	2.911	2.911	3.515	5.528	6.831	6.981
0.65	3.213	3.398	3.470	3.502	4.109	7.332	9.987	9.482
0.70	3.769	4.193	4.376	4.541	4.899	10.642	17.997	13.978

Table 2
Normalized effective moduli C_{44}^{eff}/μ_0 and C_{55}^{eff}/μ_0.

c	C_{44}^{eff}/μ_0				C_{55}^{eff}/μ_0		
	Eq. (56), non-int.	$N = 4$	$N = 9$	RUC	Eq. (57), non-int.	$N = 9$	RUC
0.05	1.089	1.086	1.085	1.084	1.105	1.105	1.105
0.10	1.187	1.176	1.173	1.169	1.221	1.221	1.221
0.15	1.298	1.272	1.264	1.257	1.351	1.351	1.351
0.20	1.421	1.375	1.362	1.349	1.497	1.499	1.497
0.25	1.562	1.487	1.468	1.447	1.662	1.666	1.663
0.30	1.722	1.613	1.584	1.556	1.851	1.860	1.854
0.35	1.906	1.755	1.716	1.679	2.069	2.087	2.076
0.40	2.122	1.919	1.867	1.821	2.322	2.358	2.340
0.45	2.376	2.113	2.046	1.991	2.622	2.688	2.659
0.50	2.680	2.348	2.265	2.200	2.980	3.105	3.058
0.55	3.052	2.643	2.540	2.473	3.418	3.651	3.578
0.60	3.515	3.029	2.905	2.846	3.963	4.410	4.294
0.65	4.109	3.568	3.425	3.406	4.662	5.563	5.372
0.70	4.899	4.406	4.265	4.393	5.590	7.609	7.274

Table 3
Normalized effective moduli C_{13}^{eff}/C_{13}^{m} and C_{33}^{eff}/C_{33}^{m}.

c	C_{13}^{eff}/C_{13}^{m}			C_{33}^{eff}/C_{33}^{m}		
	Eq. (58), non-int.	$N = 9$	RUC	Eq. (59), non-int.	$N = 9$	RUC
0.05	1.032	1.032	1.032	9.633	9.633	9.633
0.10	1.068	1.068	1.068	18.267	18.267	18.267
0.15	1.109	1.109	1.109	26.902	26.902	26.902
0.20	1.154	1.154	1.154	35.538	35.538	35.538
0.25	1.205	1.205	1.205	44.176	44.176	44.176
0.30	1.263	1.264	1.264	52.815	52.815	52.815
0.35	1.330	1.332	1.331	61.457	61.457	61.457
0.40	1.409	1.412	1.411	70.102	70.102	70.102
0.45	1.501	1.508	1.506	78.750	78.751	78.751
0.50	1.611	1.626	1.623	87.403	87.405	87.405
0.55	1.746	1.775	1.772	96.061	96.067	96.068
0.60	1.913	1.974	1.974	104.729	104.741	104.744
0.65	2.127	2.258	2.275	113.408	113.434	113.445
0.70	2.411	2.720	2.804	122.104	122.165	122.204

where $\kappa_1 = 3 - 4\nu_1$.

Substituting Eqs. (50)–(54) into Eqs. (43)–(49) and using Eqs. (39) and (40), one arrives at the following expressions for the effective stiffness coefficients

$$\frac{C_{11}^{ef} + C_{12}^{ef}}{2} = K_0 + \frac{c}{1/(K_1 - K_0) + (1-c)/(\mu_0 + K_0)} \tag{55}$$

$$\frac{C_{11}^{ef} - C_{12}^{ef}}{2} = C_{44}^{ef} = \mu_0 + \frac{c}{1/(\mu_1 - \mu_0) + (1-c)[1 + K_0/(2\mu_0)]/(\mu_0 + K_0)} \tag{56}$$

$$C_{55}^{ef} = \mu_0 + \frac{c}{1/(\mu_1 - \mu_0) + (1-c)/(2\mu_0)} \tag{57}$$

$$C_{13}^{ef} = 2K_0\left(\frac{K_0 - \mu_0}{2K_0} + \frac{c}{\mu_0/K_0 + \Lambda}\,\frac{b}{\mu_1 - a/g}\right) \tag{58}$$

$$C_{33}^{ef} = (K_0 + \mu_0)\left\{1 + \frac{2c}{ag}\left[\frac{(a+b)(a-2b)}{\mu_1\mu_0/K_0 - a} + \frac{2b^2}{\mu_0/K_0 + \Lambda}\,\frac{1}{\mu_1 - a/g}\right]\right\} \tag{59}$$

where

$$a = \mu_1\mu_0(K_1 - K_0)/(K_0K_1)$$

$$b = \mu_1\mu_0[(K_1 - K_0) - (\mu_1 - \mu_0)]/(2K_0K_1)$$

$$g = (K_0 + \mu_0)/K_0$$

$$\Lambda = -\frac{c}{g\mu_1/a - 1} \tag{60}$$

Eqs. (55)–(59) represent the effective elastic constants of a transversely isotropic material and can be obtained from the corresponding equations given by McCartney (2010) and Mogilevskaya et al. (2012b) for that material. In the latter works it was shown

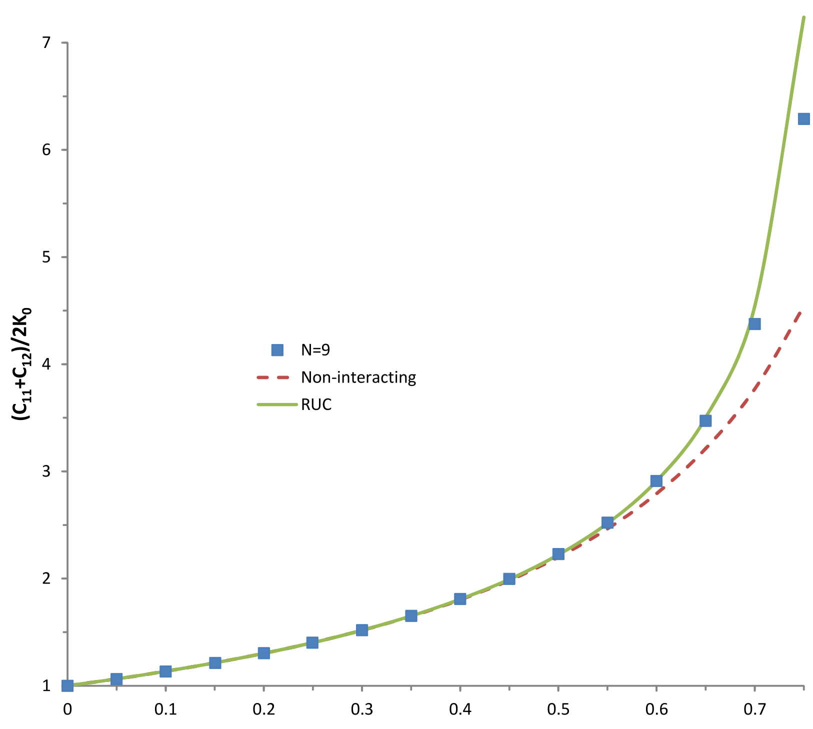

Fig. 5. Normalized effective modulus $(C_{11}^{ef} + C_{12}^{ef})/K_0$ for high contrast solid.

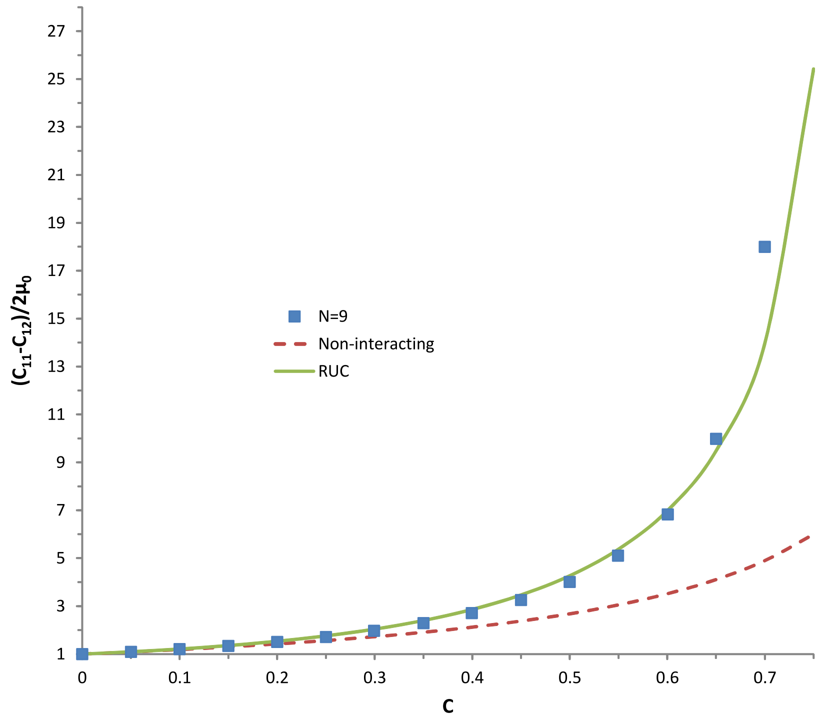

Fig. 6. Normalized effective modulus $(C_{11}^{ef} - C_{12}^{ef})/\mu_0$ for high contrast solid.

that the expressions coincide with the predictions of many well-known effective medium theories and variational bounds. In particular, the expressions given by Eqs. (55)–(59) coincide with those for one of the Hashin–Shtrikman bounds (Hashin and Shtrikman, 1963). It is clear that non-interacting estimates can not capture the overall anisotropy of tetragonal materials.

7. Numerical results

As initial tests, it was verified that, when the fibers in the cluster are arranged in such a way that the in-plane overall properties are isotropic, the expressions for the effective moduli predicted by Eqs. (43)–(49) are those of transversely isotropic elasticity. Indeed,

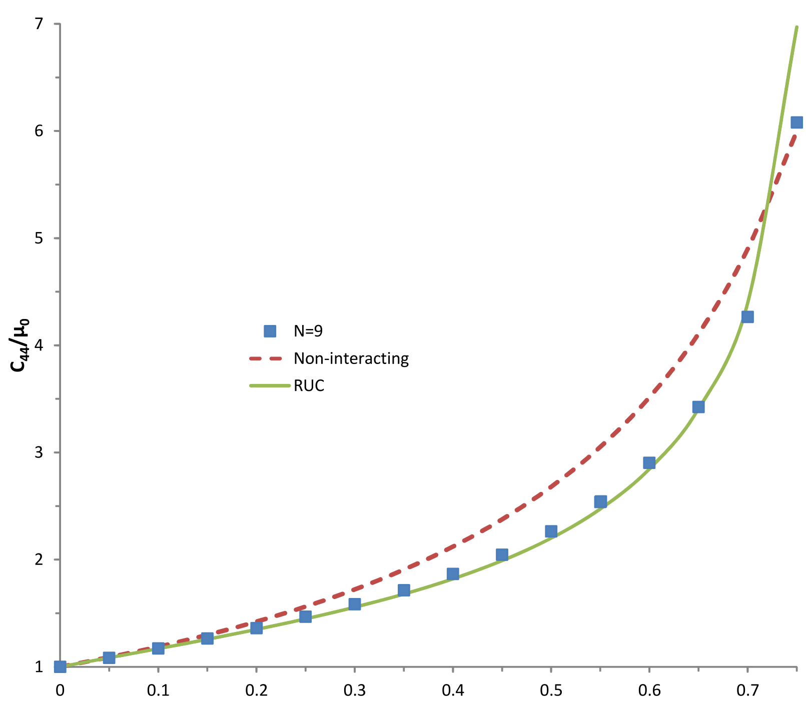

Fig. 7. Normalized effective modulus C_{44}^{ef}/μ_0 for high contrast solid.

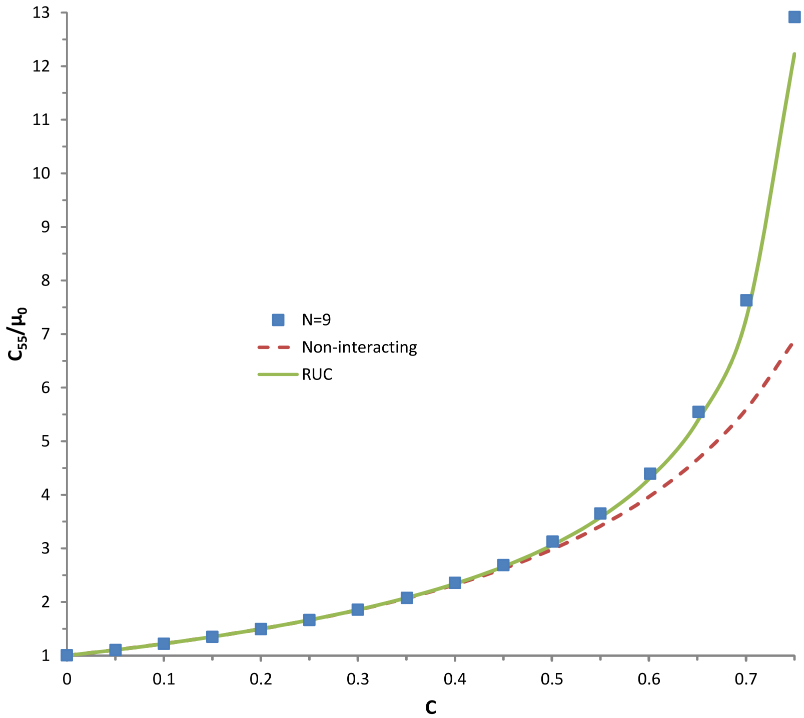

Fig. 8. Normalized effective modulus C_{55}^{ef}/μ_0 for high contrast solid.

considering the cases of fibers arranged in triangular lattice, we obtained $C_{11}^{ef}+C_{12}^{ef}=2K^{ef}$, $C_{11}^{ef}-C_{12}^{ef}=2C_{44}^{ef}=2\mu^{ef}$ and were able to reproduce all the results reported in Tables 1–7 of Section 9.1 in Mogilevskaya et al. (2012b).

In order to test if the approach can predict the overall properties of tetragonal materials, we evaluate effective elastic moduli of a composite with a square arrangement of fibers. Accurate tabulated estimates for the effective moduli of such materials are reported in Pobedrya (1984) (for some materials and for selected volume fractions of the fibers); graphical results are also given in Sabina et al. (2002).

In the following studies we adopt the same elastic constants $\nu_0=0.39$, $\nu_1=0.2$ as in Pobedrya, 1984, and consider the case of high contrast composite shear moduli $\mu_1/\mu_0=400$ to test the method under demanding conditions. The effective elastic moduli for the composite obtained with the present approach are presented in Tables 1–3 for a wide range of volume fractions c. The results for in-plane moduli are presented for two values of $N=4$ and $N=9$; for axial moduli, where the convergence is faster, only the results for $N=9$ are presented. The packing limit for the composite is $c_{\max}=\pi/4\approx 0.7854$.

Tables 1–3 contain the accurate (with relative error below 10^{-5}) values of effective moduli of the periodic fibrous composite. Obtained by the rigorous analytical method described in Kushch et al. (2008) and Kushch (2013) (by analytically solving the problem of the repetitive unit cell, RUC), these benchmark data coincide with the results reported earlier by Pobedrya (1984) for selected volume fractions. Tables 1–3 also contain the results obtained by non-interacting approximations of Eqs. (55)–(59). Graphical representations of the results for four effective moduli are provided in Figs. 5–8. The results for the remaining two axial moduli are not affected by interactions, and the graphs are not provided because the difference in different estimates would not be visible.

It can be seen that the generalized Maxwell's approach is capable of capturing the quite pronounced microstructure-induced overall anisotropy of the material. For high contrast materials, the method provides excellent estimates for the moduli up to the volume fractions $c=0.65$ but over-predicts their values for volume fractions near the packing limit. However, we believe that the same problem will exist with traditional representative volume element (RVE)-based approaches if non-periodic conditions are used on the boundary of the RVE. The results for moderate contrast materials $(\mu_1/\mu<10)$ are not shown, as they exhibit excellent agreement with exact periodic estimates up to the packing limit.

8. Conclusions

In this paper the generalized Maxwell approach based on the concept of an equivalent anisotropic inhomogeneity is applied for evaluating the overall elastic properties of tetragonal unidirectional composites with isotropic phases (matrix and fibers). The closed form expressions for the six elastic moduli that characterize its overall behavior are provided in terms of dipole coefficients for individual fibers. These closed form expressions result from comparisons of far-field solutions for the cluster with analytical solutions for a single, tetragonal, circular cylindrical equivalent inhomogeneity. To obtain complete set of constants, the analytical solution of the generalized plane problem for a circular anisotropic inhomogeneity embedded in an anisotropic matrix is derived in this paper for the first time. It is demonstrated that non-interacting estimates can not capture the overall microstructure induced anisotropy of tetragonal composite materials considered in this work. The method would work for any microstructure-induced anisotropy that is consistent with the anisotropy present in the exact solution obtained for a single equivalent inhomogeneity (6 constants or fewer, in our case). The illustrative examples involving composites with a square arrangement of fibers demonstrate that the approach provides estimates that are consistent with those predicted by the exact periodic model for a wide range of volume fractions. The approach can be used for the analysis of tetragonal materials reinforced with the fibers of arbitrary shapes, if the problem of the cluster (Section 4) is solved with the boundary element method. The investigation of the convergence of the algorithm in terms of cluster size is left to the future studies. This is a separate problem, which deserves much more attention.

Even though in this paper the effective elastic constants are expressed in terms of the dipole coefficients obtained in Section 4 using one specific approach, the formulas defining those effective constants are usable if the problem of the cluster is solved by a different approach, such as the boundary element method or finite element method. Either one of those methods is capable to provide the displacements on the surface of each fiber. Using Fourier expansions of those displacements, the dipole coefficients can be extracted with the accuracy dependent of the quality of the model employed. Those coefficients can be used in the formulas of Section 5 to obtain the effective material constants. Our numerical experiments indicate that to obtain sufficiently accurate dipole coefficients, 20 to 40 terms in the Fourier series is required.

Acknowledgement

The first author gratefully acknowledges support from the Theodore W. Bennett Chair, University of Minnesota.

Appendix A. Plane strain and antiplane problems for a single anisotropic inhomogeneity

The Stroh formalism-based solution for the problem of a single anisotropic circular inhomogeneity embedded in anisotropic matrix and subjected to the uniform far-field load provides the following expressions for the fields within the perfectly bonded inhomogeneity (see Ting, 1996 for more general case of an elliptic inhomogeneity)

$$\begin{bmatrix}\boldsymbol{\epsilon}_1^{in}\\ \mathbf{t}_2^{in}\end{bmatrix} = \left[\mathbf{I} + \left(\widetilde{\mathbf{N}}^m + \mathbf{N}^{in}\right)^{-1}\left(\mathbf{N}^m - \mathbf{N}^{in}\right)\right]\begin{bmatrix}\boldsymbol{\epsilon}_1^{\infty}\\ \mathbf{t}_2^{\infty}\end{bmatrix} \tag{A.1}$$

$$\begin{bmatrix}\boldsymbol{\epsilon}_2^{in}\\ -\mathbf{t}_1^{in}\end{bmatrix} = \mathbf{N}^{in}\begin{bmatrix}\boldsymbol{\epsilon}_1^{in}\\ \mathbf{t}_2^{in}\end{bmatrix} \tag{A.2}$$

where

$$\boldsymbol{\epsilon}_1^{\infty} = \begin{bmatrix}\varepsilon_{11}^{\infty}\\ 0\\ 2\varepsilon_{31}^{\infty}\end{bmatrix}, \quad \boldsymbol{\epsilon}_2^{\infty} = \begin{bmatrix}2\varepsilon_{12}^{\infty}\\ \varepsilon_{22}^{\infty}\\ 2\varepsilon_{32}^{\infty}\end{bmatrix} \tag{A.3}$$

$$\mathbf{t}_1^{\infty} = \begin{bmatrix}\sigma_{11}^{\infty}\\ \sigma_{21}^{\infty}\\ \sigma_{31}^{\infty}\end{bmatrix}, \quad \mathbf{t}_2^{\infty} = \begin{bmatrix}\sigma_{12}^{\infty}\\ \sigma_{22}^{\infty}\\ \sigma_{32}^{\infty}\end{bmatrix} \tag{A.4}$$

$\mathbf{t}_\mathbf{k}^{in}$, $k = 1,2$, are the vectors of the same type as those in Eqs. (A.4) but written for the stresses within the inhomogeneity instead of those at infinity, I is an identity matrix, and the vectors $\boldsymbol{\epsilon}_k^{in}$, $k = 1,2$ are

$$\boldsymbol{\epsilon}_1^{in} = \begin{bmatrix}\varepsilon_{11}^{in}\\ \omega\\ 2\varepsilon_{31}^{in}\end{bmatrix}, \quad \boldsymbol{\epsilon}_2^{in} = \begin{bmatrix}2\varepsilon_{12}^{in} - \omega\\ \varepsilon_{22}^{in}\\ 2\varepsilon_{32}^{in}\end{bmatrix} \tag{A.5}$$

with ω being the rotation of the axis of the circular inhomogeneity.

The matrices $\mathbf{N}^m$, $\mathbf{N}^{in}$ involved in Eqs. (A.1) and (A.2) represent the following matrix $\mathbf{N}$ written for the matrix and inhomogenuity, respectively,

$$\mathbf{N} = \begin{bmatrix}\mathbf{N}_1 & \mathbf{N}_2\\ \mathbf{N}_3 & \mathbf{N}_1^T\end{bmatrix} \tag{A.6}$$

where

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T, \quad \mathbf{N}_2 = \mathbf{T}^{-1}, \quad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^\mathbf{T} - \mathbf{Q} \tag{A.7}$$

and

$$T_{mn} = C_{m2n2}, \quad R_{mn} = C_{m1n2}, \quad Q_{mn} = C_{m1n1} \tag{A.8}$$

with C_{mnks} being stiffness coefficients of the corresponding material ($\sigma_{mn} = C_{mnks}\varepsilon_{ks}$) equivalent to those given in contracted notations in Eq. (2).

The matrix $\widetilde{\mathbf{N}}^m$ involved in Eq. (A.1) is related to the matrix material and expressed via the three Barnett–Lothe tensors $\mathbf{S}$, $\mathbf{H}$, and $\mathbf{L}$ as follows (see Ting, 1996):

$$\widetilde{\mathbf{N}}^m = \begin{bmatrix}\mathbf{S} & \mathbf{H}\\ -\mathbf{L} & \mathbf{S}^T\end{bmatrix} \tag{A.9}$$

For an isotropic matrix, the three tensors can be explicitly expressed as follows:

$$\mathbf{S} = \frac{\kappa_0 - 1}{\kappa_0 + 1}\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \quad \mathbf{H} = \frac{1}{\mu_0}\begin{bmatrix}\kappa_0/(\kappa_0+1) & 0 & 0\\ 0 & \kappa_0/(\kappa_0+1) & 0\\ 0 & 0 & 1\end{bmatrix},$$
$$\mathbf{L} = \mu_0\begin{bmatrix}4/(\kappa_0+1) & 0 & 0\\ 0 & 4/(\kappa_0+1) & 0\\ 0 & 0 & 1\end{bmatrix} \tag{A.10}$$

By prescribing a specific far-field load, the vectors $\mathbf{t}_k^{\infty}$, $\boldsymbol{\epsilon}_k^{\infty}$, $k = 1,2$ of Eqs. (A.3), (A.4) are found, and the uniform fields inside the inhomogeneity are expressed via its elastic stiffness coefficients by Eqs. (A.1) and (A.2). Knowing these uniform fields, the complex displacements $u^{in} = u_1^{in} + iu_2^{in}$ and tractions $\sigma^{in} = \sigma_n^{in} + i\sigma_s^{in}$ at the boundary L_{ef} (u_1^{in}, u_2^{in} are the components of the displacements in the global Cartesian coordinate system; σ_n^{in}, σ_s^{in} are normal and shear components of tractions) for the plane strain and antiplane problems can be expressed as follows

$$u^{in} = \frac{CR_{ef}}{2}\left[A_1 g_0^{-1}(z) + A_{-1}g_0(z)\right] \tag{A.11}$$

$$\sigma^{in} = -C\left[B_0 + B_{-2}g_0^2(z)\right] \tag{A.12}$$

where, as before,

$$g_0(z) = R_{ef}/z = \exp(-\imath\varphi) \tag{A.13}$$

and the coefficients $C, A_{-1}, A_1, B_0, B_{-2}$ can be expressed in terms of the elastic properties of the inhomogeneity and the load at infinity via the following relations

$$CA_1 = \varepsilon_{11}^{in} + \varepsilon_{22}^{in}, \quad CA_{-1} = \varepsilon_{11}^{in} - \varepsilon_{22}^{in} \tag{A.14}$$

$$CB_0 = -(\sigma_{11}^{in} + \sigma_{22}^{in})/2, \quad CB_{-2} = -(\sigma_{11}^{in} - \sigma_{22}^{in})/2 \tag{A.15}$$

where the corresponding fields are found from Eqs. (A.1) and (A.2).

The complex displacements at the point $z = r\exp(\imath\varphi)$ outside of the inhomogeneity (Fig. 2) could be represented by either using matrix expressions provided in Ting (1996), or (as it was done in this case) by solving a problem of an isotropic matrix containing a hole of radius R_{ef} with the displacements or tractions on its boundary prescribed by Eqs. (A.11) and (A.12).

It should be noted that an alternative solution of the inhomogeneity problem discussed in this Appendix could be based on the Lekhnitskii (1950, 1957) formalism (see also Kosmodamiansky, 1976).

Appendix B. Generalized plane strain problem for a single anisotropic inhomogeneity

As mentioned in Section 3.3, the generalized plane strain problem of a perfectly bonded inhomogeneity in an infinite matrix can be re-formulated as a standard plane strain problem of an inhomogeneity with an imperfect interface that is categorized by the conditions of continuity of displacements and a specific form of jump of tractions.

To our best knowledge, the only available solutions of plane strain problems of an anisotropic inhomogeneity imperfectly bonded to anisotropic matrix are due to Ting (2009) and Wang (2009), who considered the interface conditions that include continuity of tractions and a specific form of jump in displacements. They showed that the stresses and strains inside the inhomogeneity remain uniform.

Using the argument that the original three-dimensional generalized plane strain problem of a perfectly bonded long anisotropic fiber of elliptic shape embedded in an anisotropic matrix and subjected to uniform strain at infinity is a limiting case of an ellipsoidal anisotropic inhomogeneity embedded in the same matrix, one can conclude (Eshelby, 1961;Mura, 1982) that the stresses and strains within the fiber are uniform.

Assuming the uniformity of the elastic fields within a circular inhomogeneity and applying the same procedure as in Section 7 of Ting (2009), one can express the fields at the point $z = R_{ef}\exp(i\varphi) \in L_{ef}$ inside the inhomogeneity as follows

$$\mathbf{u}^{in} = R_{ef}[\boldsymbol{\epsilon}_1^{in}\cos\varphi + (\mathbf{N}_1^{in}\boldsymbol{\epsilon}_1^{in} + \mathbf{N}_2^{in}\mathbf{t}_2^{in})\sin\varphi] \tag{B.1}$$

$$\boldsymbol{\phi}^{in} = R_{ef}\{\mathbf{t}_2^{in}\cos\varphi + [\mathbf{N}_3^{in}\boldsymbol{\epsilon}_1^{in} + (\mathbf{N}_1^{in})^T\mathbf{t}_2^{in}]\sin\varphi\} \tag{B.2}$$

where

$$\mathbf{u}^{in} = \boldsymbol{\epsilon}_1^{in}x_1 + \boldsymbol{\epsilon}_2^{in}x_2, \quad \boldsymbol{\phi}^{in} = \mathbf{t}_2^{in}x_1 - \mathbf{t}_1^{in}x_2 \tag{B.3}$$

The same fields on the matrix side are (Ting, 2009)

$$\mathbf{u}^m = R_{ef}\left[\left(\boldsymbol{\epsilon}_1^\infty + \frac{1}{R_{ef}}\mathbf{h}\right)\cos\varphi + \left(\boldsymbol{\epsilon}_2^\infty - \frac{1}{R_{ef}}\mathbf{Sh} - \frac{1}{R_{ef}}\mathbf{Hg}\right)\sin\varphi\right] \tag{B.4}$$

$$\boldsymbol{\phi}^m = R_{ef}\left[\left(\mathbf{t}_2^\infty + \frac{1}{R_{ef}}\mathbf{g}\right)\cos\varphi + \left(-\mathbf{t}_1^\infty + \frac{1}{R_{ef}}\mathbf{Lh} - \frac{1}{R_{ef}}\mathbf{S}^T\mathbf{g}\right)\sin\varphi\right] \tag{B.5}$$

where the matrices are given by Eqs. (A.3)–(A.10) with the conditions at infinity given by Eq. (26), and the constant vectors $\boldsymbol{\epsilon}_1^{in}$, $\mathbf{t}_2^{in}$, $\mathbf{h}$, $\mathbf{g}$ should be found from the interface conditions of continuity of displacements and jump conditions of Eq. (27).

Using these conditions and equating the coefficients of cosφ and sinφ in the corresponding expressions for the displacements and tractions, the following system of equations is obtained:

$$\boldsymbol{\epsilon}_1^{in} = \boldsymbol{\epsilon}_1^\infty + \frac{1}{R_{ef}}\mathbf{h} \tag{B.6}$$

$$\mathbf{N}_1^{in}\boldsymbol{\epsilon}_1^{in} + \mathbf{N}_2^{in}\mathbf{t}_2^{in} = \boldsymbol{\epsilon}_2^\infty - \frac{1}{R_{ef}}\mathbf{Sh} - \frac{1}{R_{ef}}\mathbf{Hg} \tag{B.7}$$

$$-[\mathbf{N}_3^{in}\boldsymbol{\epsilon}_1^{in} + (\mathbf{N}_1^{in})^T\mathbf{t}_2^{in}] = \mathbf{t}_1^\infty - \frac{1}{R_{ef}}\mathbf{Lh} + \frac{1}{R_{ef}}\mathbf{S}^T\mathbf{g} - \mathbf{d}_1 \tag{B.8}$$

$$\mathbf{t}_2^{in} = \mathbf{t}_2^\infty + \frac{1}{R_{ef}}\mathbf{g} - \mathbf{d}_2 \tag{B.9}$$

where the vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are

$$\mathbf{d}_1 = \begin{bmatrix} C_{13}^{ef} - 2K_0v_0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{d}_2 = \begin{bmatrix} 0 \\ C_{13}^{ef} - 2K_0v_0 \\ 0 \end{bmatrix} \tag{B.10}$$

After some algebra, the uniform fields inside the inhomogeneity are found to be

$$\begin{bmatrix} \boldsymbol{\epsilon}_1^{in} \\ \mathbf{t}_2^{in} \end{bmatrix} = (\widetilde{\mathbf{N}}^m + \mathbf{N}^{in})^{-1}\left\{\begin{bmatrix} \boldsymbol{\epsilon}_2^\infty \\ \mathbf{d}_1 - \mathbf{t}_1^\infty \end{bmatrix} + \mathbf{N}^{in}\begin{bmatrix} -\boldsymbol{\epsilon}_1^\infty \\ \mathbf{d}_2 - \mathbf{t}_2^\infty \end{bmatrix}\right\} - \begin{bmatrix} -\boldsymbol{\epsilon}_1^\infty \\ \mathbf{d}_2 - \mathbf{t}_2^\infty \end{bmatrix} \tag{B.11}$$

The remaining fields inside the inhomogeneity can be obtained from Eq. (A.2).

Using Eqs. (A.2) and (B.11), in which the load at infinity is given by Eq. (26), one can express the complex displacements u^{in} and tractions σ^{in} at the boundary L_{ef} (on the inhomogeneity side) by Eqs. (A.11) and (A.12) with the coefficients given by Eqs. (A.14) and (A.15). The complex displacements on the matrix side remain the same as those on the inhomogeneity side, while the complex tractions on the matrix side are

$$\sigma^m = \sigma^{in} + (C_{13}^{ef} - 2K_0v_0)/2 \tag{B.12}$$

where σ^{in} is given by Eqs. (A.12) and (A.15) in which the fields inside the inhomogeneity are obtained from Eqs. (A.2) and (B.11).

The complex displacements at the point $z = r\exp(\imath\varphi)$ in the matrix (Fig. 2) can be found using the procedure similar to that described in Appendix A.

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